Cubic 4folds with Birational Fano varieties of lines

jt w/ Corey Brooke, Sarah Frei arxiv: 2410.22259.
[BFM]

also based wk C. Brooke, S. Frei, X. Qin arxiv: 2407.18904
[BFMQ] Bir'l geometry of Fano var. of lines on cubic 4folds containing pairs of cubic scrolls.

$X \subset \mathbb{P}^5$ smooth cubic fourfold.

(BE) birational equivalence.

- Many rational cubic 4folds: Pfaffian

$\mathcal{C}$ = moduli of smooth cubic 4folds
$\cup$
$\mathcal{C}_d$ Hassett divisors. parametrising cubics with more algebraic cycles.

$$H^{2,2}(X) \cap H^4(X, \mathbb{Z}) = \langle H^2, \Sigma \rangle$$
$$d = \mathrm{discr} \langle H^2, \Sigma \rangle .$$

conj irrational
$\mathcal{C}_8$ = cubics containing a plane
$\mathcal{C}_{12}$ = cubics containing a rational normal cubic scroll.
$\mathcal{C}_{20}$ = —— " —— a Veronese surface.

Conj: X is rational $\iff$ $X \in \mathcal{C}_d$, $d > 0$ not div by 4, 9, or any odd prime $\equiv 2 \bmod 3$.

Equivalences wrt derived category:

Kuznetsov: $D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle$

Kuznetsov category.

Conj: X rational $\Longleftrightarrow$ $\mathcal{A}_X \simeq D^b(\text{K3 surface})$.
Kuznetsov

Addington-Thomas
$\Longleftrightarrow$ condition on C_d.

(FM) Fourier-Mukai equivalence: X is a FM partner X' $\Longleftrightarrow$ $\mathcal{A}_X \simeq \mathcal{A}_{X'}$

Huybrechts: X has finitely many FM partners.

$X \notin \bigcup C_d$ has <u>no</u> non trivial FM partners.

Conj (Huybrechts): (FM) $\Rightarrow$ (BE).

Note: reverse false.

Evidence (Fan-Lai): $X \in C_{20}$ $\exists!$ FM partner $X' \in C_{20}$,

$$\begin{tikzcd}
 & Bl_V X \arrow[dl] \arrow[dr] & \\
V \subset X \arrow[rr, dashed] & & X' \supset V'
\end{tikzcd}$$

Today: two more examples supporting Huy' conj.

- $X \rightsquigarrow F(X)$ Fano variety of lines
  HK manifold
  $\Rightarrow$ rich bir'l geometry via Hodge theory.

$$H^2(F(X), \mathbb{Z}), q_X \leftarrow \text{Beauville-Bogomov-form.}$$

Note: $X \cong X' \iff (F(X), g) \cong (F(X'), g')$ as polarized

Possible: $F(X) \cong F(X')$ (bir'l).

$$X \not\cong X'.$$

3rd Equiv:

(BF) bir'l equiv of Fano variety of lines.

Today: examples (BF) $\Rightarrow$ (BE)

Unclear whether to expect in general?

$$\begin{tikzcd}
(BE) & (BF) \arrow[l] \arrow[d, "\text{true.}"] \arrow[dl, "\times"'] \\
(FM) \arrow[u] \arrow[ur, "\times"'] & (DE) \arrow[l]
\end{tikzcd}$$

counterexamples.

Examples: cubics $\in \mathcal{C}_{12}$ = cubics containing a cubic scroll.

§2. $X \in \mathcal{C}_{12}$

$T \subset X$ cubic scroll.

$\langle T \rangle \cap X = Y$ cubic 3 fold w/ 6 nodes.
Hassett-Tschinkel.

$Q \cap Y = T + T^\vee$.

- Debarre-Iliev-Manivel: bir'l geometry of X.

linear system of quadrics containing T:

$q : X \dashrightarrow \mathbb{P}^8$, $q(X) = Z_T$ Gushel Mukai 4fold.

$Z_T = Gr(2, V_5) \cap \mathbb{P}^8 \cap Q \subset \mathbb{P}(\Lambda^2 V_5)$

Z_T contains a plane Π, proj gives bir'l inverse.

$$\begin{tikzcd}
 & Bl_T X \cong Bl_\Pi Z_T \arrow[dl] \arrow[dr] & \\
X \arrow[rr, dashed] & & Z_T
\end{tikzcd}$$

Kuznetsov-Perry:

$\mathcal{A}_X \simeq \mathcal{A}_{Z_T}$.

Thm 1 (BFMQ): Let $X \in \mathcal{C}_{12}$ general.

$F = F(X)$.

Then F has 3 nonisomorphic bir'l HK models.

- itself
- $\widetilde{W}_T$ } double EPW sextics
- $\widetilde{W}_{T^\vee}$ } constructed from Z_T, $Z_{T^\vee}$.

double EPW sextics: O'Grady, Debarre - Iliev - Manivel, Debarre - Kuznetsov ....

Iliev-Manivel: $Z_T \subset \mathbb{P}^8 \subset \mathbb{P}(\Lambda^2 V_5)$.

$I = |\mathcal{O}_Z(2)|$ quadrics cont. Z.

$\mathrm{Disc}(Z) \subset I$ component that not restriction of Plücker quadrics.

EPW sextic 4fold.

$W = \mathrm{Disc}(Z)^\vee \subset I^\vee$ EPW sextic.

$w \in W \rightsquigarrow H_w \subset I$.

base locus quadric 3fold Q_w is singular.

$\Rightarrow \widetilde{W} \xrightarrow{\text{double cover}} W$

double EPW sextic

Sketch.

$$\begin{array}{ccc} X & \dashrightarrow & Z_T \\ \cup & & \cup \\ l & & q(l) = C_l \end{array}$$

given by quadrics containing T.

general line meet $\langle T \rangle$ in a singl pt.

$\exists !$ quadric Q_w
$\operatorname{span}(C_\ell) \subset Q_w$ quadric 3fold.

$$\begin{aligned} F &\dashrightarrow \widetilde{W}_T \\ \ell &\longmapsto (w, \operatorname{span}(C_\ell)) \end{aligned}$$

$\square$

<u>First Example.</u>

- X containing $T_1, T_2 < X$ non-homologous cubic scrolls, $T_1 \cdot T_2 = 1$.
  Such a pair of scrolls is called nosyzygetic.

$$\mathcal{M}_{\text{nonsyz}} \subset \mathcal{C}_{12}.$$

Thm 2 (BFM): let $X \in \mathcal{M}_{\text{nonsyz}}$ general, $F = F(X)$.
Then $\exists$ $X' \in \mathcal{M}_{\text{nonsyz}}$ s.t:

(BE) $X \simeq_{\text{bir}} X'$ but <u>not</u> $\cong$.

(FM) $\mathcal{A}_X \simeq \mathcal{A}_{X'}$

(BF) $F \simeq_{\text{bir}} F'$ but not isomorphic.

* They are conj irrational.

(BFMQ): we studied bir'l geometry of F.

- $\operatorname{Mov}(F) \subset \operatorname{Pos}(F) = \{x \in \underbrace{NS(F) \otimes \mathbb{R}}_{\text{rank } 3} \mid q(x) \geq 0\}$

chamber decomp. Hassett-Tschinkel. + Bayer.

$$\mathrm{Mov}(F) = \bigcup_{\substack{f: F \dashrightarrow F' \\ \text{bir'l}}} f^* \mathrm{Nef}(F')$$

F' ↩ HK model.

$H^4(X,\mathbb{Z}) \underset{H\cdot S}{\simeq} H^2(F,\mathbb{Z})$.

$X \in \mathcal{M}_{\text{nonsyz}}$, $\mathrm{Mov}(F) = \mathrm{Pos}(F)$.

In BFM, we prove $X \simeq_{\text{bir}} X'$: $X \in \mathcal{M}_{\text{nonsyz}}$, $X' \in \mathcal{M}_{\text{nonsyz}}$.

$X \dashrightarrow Z_T \quad \leftrightsquigarrow \quad F(X) \dashrightarrow \widetilde{W}_T$

$\quad\quad\quad\; Z_T \leftrightarrow Z_{T'} \quad\quad\quad\quad\quad\quad \widetilde{W}_T \cong \widetilde{W}_{T'}$

$X' \dashrightarrow Z_{T'} \quad\quad\quad\; F(X') \dashrightarrow \widetilde{W}_{T'}$

$\Rightarrow Z_T$, $Z_{T'}$ different $\Rightarrow \widetilde{W}_T \cong \widetilde{W}_{T'}$.

Debarre-Kuznetsov: Z_T, $Z_{T'}$ same fiber of period map for GM 4folds

$\Rightarrow Z_T \simeq Z_{T'}$ bir'l $\square$.

2^{nd} Example.

$X \in \mathcal{C}_8 \cap \mathcal{C}_{12}$. $P, T \subset X$

$P.T = 1$. $\rightsquigarrow$ pot irrational

Thm (BFM): $\exists\ X' \in \mathcal{C}_8 \cap \mathcal{C}_{12}$ w/

(BE) $X \simeq X'$ bir not $\cong$.

(FM) $\mathcal{A}_X \simeq \mathcal{A}_{X'}$

(BF) $F \underset{\text{bir}}{\simeq} F'$ but not $\cong$.

$q(P) = \Pi'$ plane disjoint Π.

$q : X \dashrightarrow Z_T \xrightarrow[\text{proj}\,\Pi']{} X'$ another cubic 4fold.

F and F', we show $F \simeq F'$, not isomorphic

Counterexamples:

Thm (BFM): $X \in C_{32}$. $\exists\ X' \in C_{32}$, $X \not\cong X'$ but (BF) $F(X) \cong F(X')$

$(\neg FM)$: $\mathcal{A}_X \not\cong \mathcal{A}_{X'}$.

Q: Is X and X' birational?.